Resonances in Nonstationary, Nonlinear, Multidegree-of-Freedom Systems

BRIJ N. AGRAWAL*
COMSAT Laboratories, Clarksburg, Md.
AND
R. M. EVAN-IWANOWSKI†
Syracuse University, Syracuse, N.Y.

The asymptotic method for determining resonant responses of nonstationary, nonlinear systems is presented. Resonance conditions, resonance coefficients, and higher-order resonances are discussed. The first asymptotic approximation nonstationary solution is obtained for general resonances. A gyroscopic system is analyzed for combination differential resonances $\nu = \omega_2 - 2\omega_1$ and $\nu = \omega_2 - \omega_1$. Using the general solution, nonstationary and stationary responses and stability conditions are obtained. The numerical results indicate that the change in the rate of variation of the frequency of excitation may shift the nonstationary response from one stable mode to another stable mode.

Nomenclature

A_j^m = nonoscillatory function
a_j = amplitude of the jth mode
B_j^m = nonoscillatory function
b = linear stiffness of bearing C in the α and β directions (Fig. 1)
$\bar{b}_2$ = quadratic nonlinear coefficient of bearing C in the β direction
$\bar{b}_3$ = cubic nonlinear coefficient of bearing C in the β direction
D = $\nu(\tau)[\partial/\partial\theta]+\sum_{l=1}^{n}\omega_l[\partial/\partial\psi_l]$
$\bar{e}$ = eccentricity of the rotor, D, with respect to the rotation axis (Fig. 1)
$F_{cjk}{}^m$ = coefficient of $\cos(k_0\theta+\sum_{r=1}^{n}k_r\psi_r)$ in $f_j^m = F_{cjk_0\ldots k_n}{}^m$
$F_{sjk}{}^m$ = coefficient of $\sin(k_0\theta+\sum_{r=1}^{n}k_r\psi_r)$ in $f_j^m = F_{sjk_0\ldots k_n}{}^m$
$F_{cjj}{}^1$ = coefficient of $\cos\psi_j$ in f_j^1
$F_{sjj}{}^1$ = coefficient of $\sin\psi_j$ in f_j^1
f_j = perturbation force in the jth mode
I = moment of inertia of the rotor with respect to the transverse axis passing through the rotor's c.g.
I_p = moment of inertia of the rotor with respect to the axis of symmetry
I_1 = moment of inertia of the rotor with respect to an axis passing through the lower bearing perpendicular to the symmetry axis, $I_1 = I+ML_2^2$
L = upper limit of time interval
L_1 = distance from bearing O to bearing C (Fig. 1)
L_2 = distance from bearing O to the rotor's c.g.
M = mass of the rotor
m = order of the asymptotic approximation
n = number of degrees of freedom
P = weight of the rotor
U_j^m = periodic function of angles θ and $\psi_1,\ldots,\psi_n$
X_j = normalized coordinate
X_{j0} = first asymptotic approximation of X_j, $X_{j0} = a_j\cos\psi_j$
$\dot{X}_{j0}$ = first asymptotic approximation of $\dot{X}_j$, $\dot{X}_{j0} = -a_j\omega_j\sin\psi_j$
α,β = angles defining the position of the axis of the gyroscope in the fixed coordinate axis $OXYZ$ (Fig. 1)
ε = small positive parameter
θ = phase angle of the external periodic excitation; angle of rotation of the rotor in Fig. 1
ν = instantaneous frequency of the external excitation; $\nu = \dot{\theta}$
τ = slow time, $\tau = \varepsilon t$, which varies from 0 to L
τ^* = specific time in the range of τ, $\tau^*\varepsilon[0, L]$
ψ_j = phase angle of the jth mode
Ω = angle between the axis of symmetry of the rotor system and the rotation axis
ω_j = natural frequency of the jth mode of the linear system
$\dot{}$ = d/dt, or differentiation with respect to time, t

Superscripts
$1,2,\ldots,m$ = order of the asymptotic approximation for A_j, B_j, f_j, N_j, U_j, k_r; power for the remaining symbols except g's

Subscripts
c = coefficient of the cosine function
j = jth mode
k_r = coefficient of ψ_r in the harmonic function associated with the term
k_0 = coefficient of θ in the harmonic function associated with the term
s = coefficient of the sine function

Introduction

NONSTATIONARY mechanical systems are those systems whose parameters, such as mass, stiffness, natural frequency, and external perturbation frequency, are time dependent. These systems are frequently encountered in practical applications such as transition resonance of turboengines, vibration testing of space vehicles, and variable mass of a rocket during launch.

Lewis[1] was the first to present a solution for the response of a nonstationary, linear, single-degree-of-freedom mechanical system subjected to an excitation whose frequency is a linear function of time. An outstanding contribution in this field of mechanics was also made by the Russian school. In particular, Mitropolskii extended the asymptotic method to nonstationary problems, although he did not mention combination resonances in his monograph.[2] Combination resonances and related concepts, such as resonance coefficients and resonance conditions

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* Member of the Technical Staff. Member AIAA.

† Professor, Department of Mechanical and Aerospace Engineering.

in stationary nonlinear systems, are discussed by Mettler,[3] who applied the averaging method, and by Leiss,[4] who used the asymptotic method. An exhaustive bibliography on the subject of nonstationary systems can be found in a survey paper by Evan-Iwanowski.[5] Nayfeh and Saric[6] have analyzed spinning bodies for combination resonances by using the method of multiple scales.

In this paper, the asymptotic method is presented to determine the resonant response of nonstationary, nonlinear, multidegree-of-freedom systems for general resonances such as combination resonances. The first asymptotic approximation solution is obtained for the general resonance. The concept of virtual work is applied to define resonance, resonance coefficients, and higher-order resonances. A gyroscopic system exhibiting combination differential resonances $\nu = \omega_2 - 2\omega_1$ and $\nu = \omega_2 - \omega_1$ is analyzed. The general solution is used to obtain the nonstationary response, stationary response, and stability conditions for these resonances. Nonstationary responses are obtained for the various functions of the frequency of excitation. The details of the work presented in this paper are given in Ref. 7.

Asymptotic Method

The equations of motion of an n-degree-of-freedom, asymptotic, holonomic mechanical system can be normalized and written in the following form:

$$\ddot{X}_j + \omega_j^2(\tau)X_j = \varepsilon f_j(\tau, \theta, X_1, \ldots, X_n, \dot{X}_1, \ldots, \dot{X}_n) \qquad j = 1, \ldots, n \tag{1}$$

In Eq. (1), the terms which are functions of τ are varying slowly with time. The method presented in this paper requires that the system parameters vary slowly compared to a natural time unit, which is a time unit of the order of the vibration period. The time τ varies from 0 to L. Setting $\varepsilon = 0$ in Eq. (1) and assuming that τ is a parameter results in an equation, called an unperturbed equation, which can be solved as follows:

$$X_j = a_j \cos\psi_j, \quad \dot{a}_j = 0, \quad \dot{\psi}_j = \omega_j \qquad j = 1, \ldots, n \tag{2}$$

When $\varepsilon \neq 0$, i.e., in the presence of perturbation, higher harmonics may appear in the solutions and the natural frequency may depend on the amplitude. Furthermore, various resonances may take place, and the variation of $\omega_j(\tau)$ and $\nu(\tau)$ with slow time, τ, will result in additional phenomena which are not observed in nonlinear stationary systems. Taking into account these physical arguments and keeping in mind that when $\varepsilon \to 0$ the solution should be represented by Eq. (2), we use the following form to solve Eq. (1) for the mth approximation:

$$X_j = a_j(\tau)\cos\psi_j(\tau) + \sum_{i=1}^{m} \varepsilon^i \times U_j^i(\tau, a_1, \ldots, a_n, \theta, \psi_1, \ldots, \psi_n) \tag{3}$$

where U_j^i are unknown functions, periodic in θ and $\psi_1, \ldots, \psi_n$ and dependent on $a_1, \ldots, a_n$. The functions a_j and ψ_j are determined from the following equations:

$$\dot{a}_j = \sum_{i=1}^{m} \varepsilon^i A_j^i(\tau, a_1, \ldots, a_n, \theta, \psi_1, \ldots, \psi_n) \tag{4a}$$

$$\dot{\psi}_j = \omega_j + \sum_{i=1}^{m} \varepsilon^i B_j^i(\tau, a_1, \ldots, a_n, \theta, \psi_1, \ldots, \psi_n) \tag{4b}$$

where A_j^i and B_j^i are nonoscillatory functions.

U_j^i, A_j^i, and B_j^i are selected so that, after a_j and ψ_j are replaced with the functions defined in Eq. (4), Eq. (3) will satisfy Eq. (1) up to ε^m. The coefficients A_j^i and B_j^i are also unknowns in the determination of X_j. Obviously, Eq. (1) is insufficient to determine the unique values of these coefficients. To obtain unique values, an additional condition is necessary; i.e., U_j^i must be finite.

The asymptotic method presented here is similar to the asymptotic method developed by Mitropolskii for nonstationary systems. The essential difference lies in the form in which the solution is sought. In the present method, this form is the same for all resonances; in Mitropolskii's method, it changes for different resonances. The former approach, as will be clear later, is a unified approach for all resonances, and makes it possible to obtain resonance coefficients and conditions. Expanding f_j in the right-hand side of Eq. (1) into Taylor's series results in

$$\varepsilon f_j(\tau, \theta, X_1, \ldots, X_n, \dot{X}_1, \ldots, \dot{X}_n) = \sum_{i=1}^{\infty} \varepsilon^i f_j^i(\tau, \theta, a_1, \ldots, a_n, \psi_1, \ldots, \psi_n) \tag{5}$$

where

$$\begin{aligned} f_j^1 &= f_j(\tau, \theta, X_{10}, \ldots, X_{n0}, \dot{X}_{10}, \ldots, \dot{X}_{n0}) \\ X_{i0} &= a_i \cos\psi_i \\ \dot{X}_{i0} &= -a_i \omega_i \sin\psi_i \end{aligned} \tag{6}$$

After determining the first and second derivatives of X_j with respect to time t by using Eqs. (3) and (4) and substituting X_j and $\ddot{X}_j$ in the left-hand side of Eq. (1), we equate the coefficients of the same power of ε, up to an including mth-order terms, in the left and right sides of Eq. (1) to obtain

$$\begin{aligned} D^2U_j^1 + \omega_j^2 U_j^1 = {} & \sin\psi_j[a_j(\partial\omega_j/\partial\tau) + 2A_j^1\omega_j + a_j DB_j^1] - \\ & \cos\psi_j(DA_j^1 - 2a_j B_j^1 \omega_j) + \\ & f_j(\tau, \theta, X_{10}, \ldots, X_{n0}, \dot{X}_{10}, \ldots, \dot{X}_{n0}) \end{aligned} \tag{7a}$$

$$\vdots$$

$$\begin{aligned} D^2U_j^m + \omega_j^2 U_j^m = {} & \sin\psi_j(2A_j^m\omega_j + a_j DB_j^m) - \\ & \cos\psi_j(DA_j^m - 2a_j B_j^m \omega_j) + f_j^m - N_j^m \end{aligned} \tag{7b}$$

where the differential operators D and N_j^m are defined as follows:

$$D = \nu(\tau)(\partial/\partial\theta) + \sum_{l=1}^{n} \omega_l(\partial/\partial\psi_l) \tag{8a}$$

$$N_j^m = N_j^m(A_j^1, \ldots, A_j^{m-1}, B_j^1, \ldots, B_j^{m-1}, U_j^1, \ldots, U_j^{m-1}) \tag{8b}$$

The steps leading to the mth-order approximation are as follows. Calculate U_j^1, A_j^1, and B_j^1 by solving Eq. (7a) and constraining U_j^1 to exclude secular terms. In a similar manner, for the mth approximations, the values of U_j^i, A_j^i, and B_j^i $(i = 1, 2, \ldots, m-1)$ obtained from the previous steps are substituted into $f_j^m - N_j^m$ in Eq. (7b). U_j^m, A_j^m, and B_j^m are then calculated by solving Eq. (7b) and constraining U_j^m to exclude secular terms. Substituting U_j^m, A_j^m, and B_j^m into Eqs. (3) and (4) yields the mth asymptotic solution.

Resonances

Resonance is characterized by a large system response amplitude caused by a small perturbation force. This phenomenon can be explained in terms of virtual work; that is, it takes place when the virtual work done by the perturbing forces over a cycle of a particular mode is not equal to zero over a large time interval.

Consider the virtual work of the perturbing force εf_j along the virtual displacement corresponding to the mode of the first harmonic of X_j; i.e.,

$$\text{virtual work} = \varepsilon f_j \delta X_j = \sum_{m=1}^{\infty} \varepsilon^m [f_j^m(\delta a_j \cos\psi_j - \delta\psi_j a_j \sin\psi_j)] \tag{9}$$

Expanding f_j^m into Fourier series results in

$$f_j^m = \sum_{k_0} \cdots \sum_{k_n} \left[F_{cjk_0\ldots k_n}{}^m \cos\left(k_0\theta + \sum_{r=1}^{n} k_r\psi_r\right) + F_{sjk_0\ldots k_n}{}^m \sin\left(k_0\theta + \sum_{r=1}^{n} k_r\psi_r\right)\right] \tag{10}$$

Henceforth, $F_{cjk_0\ldots k_n}$ and $F_{sjk_0\ldots k_n}$ will be referred to as F_{cjk} and F_{sjk}, respectively. Substituting f_j^m from Eq. (10) into Eq. (9) and averaging the virtual work over a large time interval, T, indicates that only nonperiodic terms will be nonzero. Hence, only those terms whose frequencies are equal to ω_j, i.e., whose indices k satisfy the following relationship:

$$k_0 v(\tau^*) + \sum_{r=1}^{n} k_r \omega_r(\tau^*) = \pm\omega_j(\tau^*) \tag{11}$$

for some time $\tau^* \varepsilon [0, L]$, will contribute to virtual work.

The Fourier coefficients F_{sjk} and F_{cjk}, which correspond to the previous resonance relationship, are called resonance coefficients. Hence, in order to have resonance, two conditions should be satisfied. First, the resonance relationship of Eq. (11) must be satisfied, and second, at least one of the corresponding resonance coefficients should be nonzero.

Clearly the resonance conditions may be satisfied by asymptotic approximations of various orders of ε. That is,

$$k_0{}^1 v + \sum_{r=1}^{n} k_r{}^1 \omega_r = \pm\omega_j$$

$$F_{cjk}{}^1 \neq 0 \quad \text{or} \quad F_{sjk}{}^1 \neq 0 \tag{12a}$$

$$\vdots$$

$$k_0{}^l v + \sum_{r=1}^{n} k_r{}^l \omega_r = \pm\omega_j$$

$$F_{cjk}{}^l \neq 0 \quad \text{or} \quad F_{sjk}{}^l \neq 0 \tag{12b}$$

where the superscripts indicate the order of ε of the asymptotic approximation.

Some of the resonances may be satisfied in more than one order of ε. Hence, the lth-order resonance may be defined as follows. Let us denote the elements of the sets of indices satisfying the lth-order resonance relationship as $\{k_r{}^l\}$. If $\{k_r{}^l\}$ is not contained in any $\{k_r{}^i\}$ where $i < l$, and if either the $F_{cjk}{}^l$ or $F_{sjk}{}^l$ or both are nonzero, then the conditions are satisfied for the existence of the lth-order resonance. This resonance relationship may be expressed as

$$k_0{}^l v + \sum_{r=1}^{n} k_r{}^l \omega_r = \pm\omega_j \tag{13}$$

The resonance relationship expressed by Eq. (11) may be rewritten as

$$h_0 v(\tau^*) = \sum_{r=1}^{n} h_r \omega_r(\tau^*) \tag{14}$$

where

$$h_0 = k_0, \quad h_r = -k_r \pm \delta_{rj}$$

The resonance expressed by Eq. (14) may be considered to be a general resonance. Other types of resonance, which are special cases of Eq. (14), are listed in Table 1.

Table 1 Resonance types

Resonance relationship		Type of resonance
$h_0 v = \sum_{r=1}^{n} h_r \omega_r$,		
	$h_r \geqq 0$	Combination additive resonance
	some $h_r < 0$	Combination differential resonance
$\sum_{r=1}^{n} h_r \omega_r = 0$, $\quad h_0 = 0$		Internal resonance
$v = \omega_j$, $\quad h_0 = 1$, $\quad h_r = \delta_{rj}$		Principal resonance
$v/h_j = \omega_j$,		
$h_0 = 1$, $\quad h_r = 0$, $\quad r \neq j$		Subharmonic resonance
$h_j = 2$		Parametric resonance
$h_0 v = \omega_j$, $\quad h_r = \delta_{rj}$		Superharmonic resonance
$h_0 v/h_j = \omega_j$		Rational resonances

First Asymptotic Approximation Solution

By expanding $f_j{}^1$ into a Fourier series and substituting the resulting values into Eq. (7a), we obtain

$$D^2 U_j{}^1 + \omega_j{}^2 U_j{}^1 = \sin\psi_j[a_j(\partial\omega_j/\partial\tau) + 2A_j{}^1\omega_j + a_j DB_j{}^1] - \cos\psi_j(DA_j{}^1 - 2a_j B_j{}^1\omega_j) + \sum_{k_0}\cdots\sum_{k_n}\left[F_{sjk}{}^1 \sin\left(k_0\theta + \sum_{r=1}^{n} k_r\psi_r\right) + F_{cjk}{}^1 \cos\left(k_0\theta + \sum_{r=1}^{n} k_r\psi_r\right)\right] \tag{15}$$

For $U_j{}^1$ to be finite, the right-hand side of Eq. (15) must not contain harmonics of frequency ω_j. Thus the terms containing harmonics of ω_j or secular terms should be set equal to zero. It should be noted that, in $f_j{}^1$, the harmonic terms whose frequency is ω_j for $\tau^* \varepsilon [0, L]$ contribute virtual work in Eq. (9); hence, these terms cause resonance. These same terms are secular terms in Eq. (15). A resonant case is discussed in the following paragraphs.

A system is resonant if both resonance conditions are satisfied. This indicates that $f_j{}^1$ contains harmonic terms whose frequency is ω_j for $\tau^* \varepsilon [0, L]$. Hence,

$$k_0 v(\tau^*) + \sum_{r=1}^{n} k_r \omega_r(\tau^*) = \pm\omega_j(\tau^*) \tag{16}$$

Equating to zero coefficients containing harmonics whose frequency is ω_j for τ^* results in

$$a_j(\partial\omega_j/\partial\tau) + 2A_j{}^1\omega_j + a_j DB_j{}^1 + F_{sjj}{}^1 \pm F_{sjk}{}^1 \cos\left(k_0\theta + \sum_{r=1}^{n} k_r\psi_r \mp \psi_j\right) \mp F_{cjk}{}^1 \sin\left(k_0\theta + \sum_{r=1}^{n} k_r\psi_r \mp \psi_j\right) = 0$$

$$DA_j{}^1 - 2a_j B_j{}^1\omega_j - F_{cjj}{}^1 - F_{sjk}{}^1 \sin\left(k_0\theta + \sum_{r=1}^{n} k_r\psi_r \mp \psi_j\right) - F_{cjk}{}^1 \cos\left(k_0\theta + \sum_{r=1}^{n} k_r\psi_r \mp \psi_j\right) = 0 \tag{17}$$

Solving Eq. (17) for $A_j{}^1$ and $B_j{}^1$ and substituting these values into Eq. (4) yields

$$\dot{a}_j = \varepsilon\left\{-\tfrac{1}{2}(F_{sjj}{}^1/\omega_j) - \tfrac{1}{2}a_j[(\partial\omega_j/\partial\tau)/\omega_j] - F_{sjk}{}^1 \times \left[\cos\left(k_0\theta + \sum_{r=1}^{n} k_r\psi_r \mp \psi_j\right)\Big/\left(k_0 v + \sum_{r=1}^{n} k_r\omega_r \pm \omega_j\right)\right] + F_{cjk}{}^1 \times \left[\sin\left(k_0\theta + \sum_{r=1}^{n} k_r\psi_r \mp \psi_j\right)\Big/\left(k_0 v + \sum_{r=1}^{n} k_r\omega_r \pm \omega_j\right)\right]\right\}$$

$$\dot{\psi}_j = \omega_j + \varepsilon\left\{-\tfrac{1}{2}(F_{cjj}{}^1/a_j\omega_j) \mp F_{sjk}{}^1 \times \left[\sin\left(k_0\theta + \sum_{r=1}^{n} k_r\psi_r \mp \psi_j\right)\Big/a_j\left(k_0 v + \sum_{r=1}^{n} k_r\omega_r \pm \omega_j\right)\right] \mp F_{cjk}{}^1\left[\cos\left(k_0\theta + \sum_{r=1}^{n} k_r\psi_r \mp \psi_j\right)\Big/a_j\left(k_0 v + \sum_{r=1}^{n} k_r\omega_r \pm \omega_j\right)\right]\right\} \tag{18}$$

It should be noted that the harmonic terms in $f_j{}^1$ whose frequency is ω_j for τ^* contribute virtual work in Eq. (9), resulting in resonance. These terms, which are secular terms in Eq. (15), contribute terms in Eq. (18).

The method of varying parameters,[8] also known as the averaging method, can also be applied to construct the approximate solution for nonstationary systems, although it must be modified so that the slowly varying parameters are regarded as constants during the averaging. For the first asymptotic approximation solution, the analysis of the averaging method is simpler than the averaging method. However, for higher-order resonances and higher-order approximation solutions, the asymptotic method presented in this paper gives a more unified approach than the averaging method.

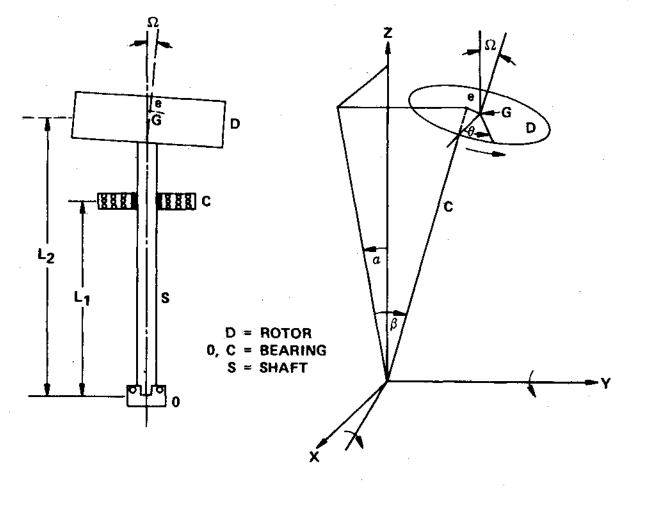

Fig. 1 Schematic representation of a gyroscopic system consisting of a disk mounted on the shaft.

Gyroscopic System

Consider the gyroscopic system shown in Fig. 1. It consists of a rotor D, mounted on the shaft, which is supported by two bearings C and O. The rigidity of the upper bearing C is only assumed to be nonlinear with respect to angle β. This assumption is made to simplify the analysis, since the resonance phenomena which will be discussed will be present even if bearing C is also nonlinear with respect to angle α. However, in this case, the analysis will be much more involved. The rotor D is assumed to be unbalanced statically and dynamically.

The differential equations of motion of the rotor D are

$$I_1\ddot{\alpha}+I_p\dot{\theta}\dot{\beta}+b_1\alpha+\bar{c}\dot{\alpha}=[(I_p-I)\bar{\Omega}+ML_2\bar{e}]\times [-\dot{\theta}^2\sin\theta+\ddot{\theta}\cos\theta]-\bar{e}P\sin\theta-I_p\ddot{\theta}\beta$$
$$I_1\ddot{\beta}-I_p\dot{\theta}\dot{\alpha}+b_1\beta+\bar{b}_2\beta^2+\bar{b}_3\beta^3+\bar{c}\dot{\beta}=[(I_p-I)\bar{\Omega}+ML_2\bar{e}]\times [\dot{\theta}^2\cos\theta+\ddot{\theta}\sin\theta]-\bar{e}P\cos\theta+I_p\ddot{\theta}\alpha \tag{19}$$

where

$$I_1=I+ML_2{}^2,\quad b_1=bL_1{}^2-PL_2$$

The bared terms are small and of the order of ε. By defining

$$\bar{b}_2=\varepsilon b_2,\quad \bar{b}_3=\varepsilon b_3,\quad \bar{c}=\varepsilon c,\quad \bar{e}=\varepsilon e$$
$$\dot{\theta}=v(\tau),\quad \ddot{\theta}=\varepsilon[\partial v(\tau)/\partial\tau],\quad \bar{\Omega}=\varepsilon\Omega \tag{20}$$

substituting Eq. (20) into Eq. (19), and neglecting terms of a higher order of ε than unity, we obtain

$$\ddot{\alpha}+(I_pv/I_1)\dot{\beta}+K\alpha=\varepsilon\{-\delta\dot{\alpha}-F_1\sin\theta-[I_p(\partial v/\partial\tau)/I_1]\beta\} \tag{21a}$$
$$\ddot{\beta}-(I_pv/I_1)\dot{\alpha}+K\beta=\varepsilon\{-\delta\dot{\beta}-K_1\beta^2-K_2\beta^3+F_2\cos\theta+[I_p(\partial v/\partial\tau)/I_1]\alpha\} \tag{21b}$$

where

$$K=b_1/I_1,\quad K_1=b_2/I_1,\quad K_2=b_3/I_1,\quad \delta=c/I_1$$
$$F_1=\{[(I_p-I)\Omega+ML_2e]v^2+eP\}/I_1 \tag{22}$$
$$F_2=\{[(I_p-I)\Omega+ML_2e]v^2-eP\}/I_1$$

Normalization

Let the solution of Eq. (21) be in the following form:

$$\alpha=\sum_{j=1}^{2}C_jY_j$$
$$\beta=\sum_{j=1}^{2}x_j \tag{23}$$

where Y_j are indefinite integrals of x_j. If Eq. (21) is unperturbed, i.e., if $\varepsilon=0$, x_j is assumed to be a harmonic function of frequency ω_j. Differentiating Eq. (21a) with respect to t and substituting α and β from Eq. (23) into the resulting equation and Eq. (21b), results in the following characteristic determinant:

$$\begin{vmatrix} K-\omega^2 & -(I_pv/I_1)\omega^2 \\ -(I_pv/I_1) & K-\omega^2 \end{vmatrix}=0 \tag{24}$$

Denoting the roots of Eq. (24) as

$$\omega=\pm\omega_j\qquad j=1,2 \tag{25a}$$

we obtain

$$\omega_1=\tfrac{1}{2}\{(I_p/I_1)v-[(I_p{}^2/I_1{}^2)v^2+4K]^{1/2}\}$$
$$\omega_2=\tfrac{1}{2}\{(I_p/I_1)v+[(I_p{}^2/I_1{}^2)v^2+4K]^{1/2}\}$$
$$C_j=-\omega_j\qquad j=1,2 \tag{25b}$$

Here the modes corresponding to the positive sign in Eq. (25a) are considered. ω_1, which is negative, represents an inverse precession, and ω_2, which is positive, represents a direct precession. By differentiating Eq. (21a) with respect to t, substituting α and β from Eq. (23) into the resulting equation and Eq. (21b), and solving for $\ddot{x}_j+\omega_j{}^2x_j$, we obtain

$$\ddot{x}_j+\omega_j{}^2x_j=\varepsilon\left[-\sum_{i=1}^{2}\lambda_{ji}\dot{x}_i-\sum_{i=1}^{2}\sum_{k=1}^{2}C_{j2}x_ix_k-\sum_{i=1}^{2}\sum_{k=1}^{2}\sum_{l=1}^{2}C_{j3}x_ix_kx_l+P\cos\theta\right]\qquad j=1,2 \tag{26}$$

where

$$\lambda_{ji}=(-1)^j[(\omega_1\omega_2I_p/I_1)(\partial v/\partial\tau)+\delta K\omega_j-\delta\omega_1\omega_2\omega_i-(K\omega_jI_p/I_1\omega_i)(\partial v/\partial\tau)]/K(\omega_2-\omega_1)$$
$$C_{j2}=(-1)^j[\omega_jK_1/(\omega_2-\omega_1)] \tag{27}$$
$$C_{j3}=(-1)^j[\omega_jK_2/(\omega_2-\omega_1)]$$
$$P_j=(-1)^j[(\omega_1\omega_2vF_1+K\omega_jF_2)/K(\omega_2-\omega_1)]$$

Asymptotic Solution

Let us assume that $v\neq\omega_1$ or ω_2 for any $\tau^*\varepsilon[0,L]$, i.e., that there is no main resonance. In this case, x_j can be represented as follows:

$$x_j=X_j+A_j(\tau)\cos\theta \tag{28}$$

where $A_j\cos\theta$ is the forced vibration of the linear system due to P_j, and A_j is given by

$$A_j=[\varepsilon P_j/(\omega_j{}^2-v^2)] \tag{29}$$

Substituting x_j from Eq. (28) and A_j from Eq. (29) into Eq. (26) and including terms up to the order of ε, results in

$$\ddot{X}_j+\omega_j{}^2X_j=\varepsilon[g_j{}^1+g_j{}^2\cos\theta+g_j{}^3\sin\theta+g_j{}^4\cos 2\theta+g_j{}^5\cos 3\theta+g_{j1}{}^6\dot{X}_1+g_{j2}{}^6\dot{X}_2+(g_{j1}{}^1+g_{j1}{}^2\cos\theta+g_{j1}{}^3\cos 2\theta)(X_1+X_2)+(g_{j2}{}^1+g_{j2}{}^2\cos\theta)(X_1+X_2)^2+g_{j3}(X_1+X_3)^3] \tag{30}$$

where

$$g_j{}^1=-\tfrac{1}{2}C_{j2}(A_1+A_2)^2$$
$$g_j{}^2=-\tfrac{3}{4}C_{j3}(A_1+A_2)^2$$
$$g_j{}^3=v(\lambda_{j1}A_1+\lambda_{j2}A_2)+2(\partial A_j/\partial\tau)v+A_j(\partial v/\partial\tau)$$
$$g_j{}^4=-\tfrac{1}{2}C_{j2}(A_1+A_2)^2$$
$$g_j{}^5=-\tfrac{1}{4}C_{j3}(A_1+A_2)^2$$
$$g_{j1}{}^6=-\lambda_{ji} \tag{31}$$
$$g_{j1}{}^1=-\tfrac{3}{2}C_{j3}(A_1+A_2)^2$$
$$g_{j1}{}^3=-\tfrac{3}{2}C_{j3}(A_1+A_2)^2$$
$$g_{j2}{}^1=-C_{j2}$$
$$g_{j2}{}^2=-3C_{j3}(A_1+A_2)$$
$$g_{j3}=-C_{j3}$$
$$g_{j1}{}^2=-2C_{j2}(A_1+A_2)$$

Equation (30) is a special case of Eq. (1) with $n=2$. Confining our analysis to the first asymptotic approximation, from Eq. (6) we obtain

$$\begin{aligned} f_j^{\,1} &= f_j(\tau,\theta,X_{10},X_{20},\dot{X}_{10},\dot{X}_{20}) \\ &= g_j^{\,1}+g_j^{\,2}\cos\theta+g_j^{\,3}\sin\theta+g_j^{\,4}\cos 2\theta+g_j^{\,5}\cos 3\theta- \\ &\quad g_{j1}{}^6 a_1\omega_1\sin\psi_1-g_{j2}{}^6 a_2\omega_2\sin\psi_2+ \\ &\quad (g_{j1}{}^1+g_{j1}{}^2\cos\theta+g_{j1}{}^3\cos 2\theta)(a_1\cos\psi_1+a_2\cos\psi_2)+ \\ &\quad (g_{j2}{}^1+g_{j2}{}^2\cos\theta)(a_1\cos\psi_1+a_2\cos\psi_2)^2+ \\ &\qquad g_{j3}(a_1\cos\psi_1+a_2\cos\psi_2)^3 \end{aligned} \tag{32}$$

Combination Differential Resonance $\nu = -2\omega_1+\omega_2$

Assume that

$$\nu(\tau^*) = \omega_2(\tau^*)-2\omega_1(\tau^*) \tag{33}$$

for some time τ^*. The terms in $f_1{}^1$ and $f_2{}^1$ which cause the resonance expressed by Eq. (33) are $\frac{1}{2}[g_{12}{}^2 a_1 a_2\cos(\theta+\psi_1-\psi_2)]$ for the resonance relationship $\nu+\omega_1-\omega_2 = -\omega_1$, and $\frac{1}{4}[g_{22}{}^2 a_1{}^2\cos(\theta+2\psi_1)]$ for the resonance relationship $\nu+2\omega_1 = \omega_2$, respectively. Equation (18) can be used to obtain the following nonstationary solution for the resonance relationship $\nu = \omega_2-2\omega_1$:

$$\begin{aligned} \dot{a}_1 &= \varepsilon\{\tfrac{1}{2}g_{11}{}^6 a_1-\tfrac{1}{2}a_1[(\partial\omega_1/\partial\tau)/\omega_1]+ \\ &\qquad \tfrac{1}{2}g_{12}{}^2 a_1 a_2[\sin(\theta+2\psi_1-\psi_2)/(\nu-\omega_2)]\} \end{aligned} \tag{34a}$$

$$\begin{aligned} \dot{\psi}_1 &= \omega_1+\varepsilon\{-[(g_{11}{}^1/2\omega_1)+3g_{13}(a_1{}^2/8\omega_1)+3g_{13}(a_2{}^2/4\omega_1)]+ \\ &\qquad \tfrac{1}{2}g_{12}{}^2 a_2[\cos(\theta+2\psi_1-\psi_2)/(\nu-\omega_2)]\} \end{aligned} \tag{34b}$$

$$\begin{aligned} \dot{a}_2 &= \varepsilon\{\tfrac{1}{2}g_{22}{}^6 a_2-\tfrac{1}{2}a_2[(\partial\omega_2/\partial\tau)/\omega_2]+ \\ &\qquad \tfrac{1}{4}g_{22}{}^2 a_1{}^2[\sin(\theta+2\psi_1-\psi_2)/(\nu+2\omega_1+\omega_2)]\} \end{aligned} \tag{34c}$$

$$\begin{aligned} \dot{\psi}_2 &= \omega_2+\varepsilon\{-[(g_{21}{}^1/2\omega_2)+3g_{23}(a_2{}^2/8\omega_2)+3g_{23}(a_1{}^2/4\omega_2]- \\ &\qquad g_{22}{}^2 a_1{}^2[\cos(\theta+2\psi_1-\psi_2)/4a_2(\nu+2\omega_1+\omega_2)]\} \end{aligned} \tag{34d}$$

In the stationary mode, amplitudes a_1 and a_2 are constant; i.e.,

$$\dot{a}_1 = \dot{a}_2 = 0 \tag{35}$$

For the resonance region, using Eqs. (34) and (35), we obtain

$$a_1{}^2/a_2{}^2 = -2(g_{22}{}^6 g_{12}{}^2\omega_2/g_{11}{}^6 g_{22}{}^2\omega_1) \tag{36a}$$

$$\begin{aligned} \nu &= \omega_2-2\omega_1+\varepsilon[(g_{11}{}^1/\omega_1)-(g_{21}{}^1/2\omega_2)+a_1{}^2\{(3g_{13}/4\omega_1)- \\ &\quad (3g_{23}/4\omega_2)-[(3g_{13}/2\omega_1)-(3g_{23}/8\omega_2)]\times \\ &\quad (g_{11}{}^6 g_{22}{}^2\omega_1/2g_{22}{}^6 g_{12}{}^2\omega_2)\}\mp \\ &\quad \{-(g_{22}{}^2 g_{12}{}^2 a_1{}^2/32\omega_1\omega_2)-(g_{22}{}^6 g_{11}{}^6/4)\}\times \\ &\qquad \{2(g_{11}{}^6/g_{22}{}^6)^{1/2}+(g_{22}{}^6/g_{11}{}^6)^{1/2}\}] \end{aligned} \tag{36b}$$

From the Routh-Hurwitz stability criteria, the stability conditions are

$$\pm(\partial\nu/\partial a_j) > 0, \quad j = 1,2 \tag{37}$$

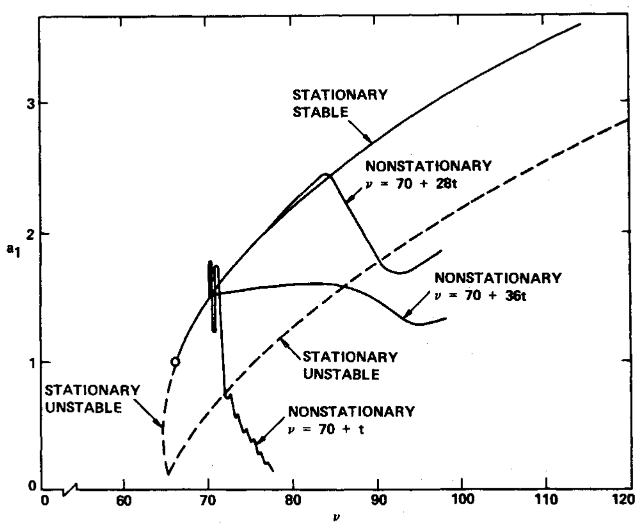

Fig. 2 Nonstationary response for a combination differential resonance, $\nu = \omega_2-2\omega_1$, for linearly increasing frequency of perturbation.

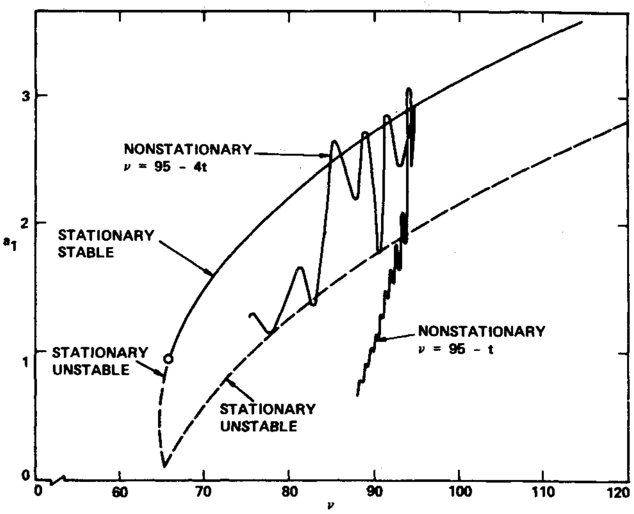

Fig. 3 Nonstationary response for a combination differential resonance, $\nu = \omega_2-2\omega_1$, for linearly decreasing frequency of perturbation.

and the stationary amplitude a_1 should be greater than $a_1{}^*$ which is the solution of the following equation:

$$\begin{aligned} &-\tfrac{1}{2}(g_{11}{}^6+g_{22}{}^6)\{(g_{11}{}^6 g_{22}{}^6/4)+ \\ &\quad [-(g_{22}{}^2 g_{12}{}^2 a_1{}^2/32\omega_1\omega_2)- \\ &\quad (g_{22}{}^6 g_{11}{}^6/4)][(g_{22}{}^6/g_{11}{}^6)-4]\}+ \\ &\quad g_{22}{}^2 g_{12}{}^2 a_1{}^2[(2g_{11}{}^6+g_{22}{}^6)/64\omega_1\omega_2]\mp \\ &\quad 2a_1{}^2[-(g_{22}{}^2 g_{12}{}^2 a_1{}^2/32\omega_1\omega_2)- \\ &\quad (g_{22}{}^6 g_{11}{}^6/4)]^{1/2}(4\omega_1/g_{12}{}^2)(g_{11}{}^6/g_{22}{}^6)^{1/2}\times \\ &\quad [-(3g_{22}{}^6 g_{22}{}^2 g_{13}/16\omega_1\omega_3)+(3g_{22}{}^6 g_{22}{}^2 g_{23}/64\omega_2{}^2)+ \\ &\quad (3g_{11}{}^6 g_{12}{}^2 g_{13}/32\omega_1{}^2)-(3g_{11}{}^6 g_{12}{}^2 g_{23}/32\omega_1\omega_2)] = 0 \end{aligned} \tag{38}$$

Combination Differential Resonance $\nu = \omega_2-\omega_1$

Assume that

$$\nu(\tau^*) = \omega_2(\tau^*)-\omega_1(\tau^*) \tag{39}$$

for some time $\tau^*\varepsilon[0,L]$. The terms in $f_1{}^1$ and $f_2{}^1$ which cause the resonance expressed by Eq. (39) are $\frac{1}{2}[g_{11}{}^2 a_2\cos(\theta-\psi_2)]$ for the resonance relationship $\nu-\omega_2 = -\omega_1$, and $\frac{1}{2}[g_{21}{}^2 a_1\cos(\theta+\psi_1)]$ for the resonance relationship $\nu+\omega_1 = \omega_2$, respectively. Equation (18) can be used to obtain the following nonstationary solution for $\nu = \omega_2-\omega_1$:

$$\begin{aligned} \dot{a}_1 &= \varepsilon\{\tfrac{1}{2}g_{11}{}^6 a_1-\tfrac{1}{2}a_1[(\partial\omega_1/\partial\tau)/\omega_1]+ \\ &\qquad \tfrac{1}{2}[g_{11}{}^2 a_2\sin(\theta-\psi_2+\psi_1)/(\nu-\omega_2-\omega_1)]\} \end{aligned} \tag{40a}$$

$$\begin{aligned} \dot{\psi}_1 &= \omega_1+\varepsilon\{-[\tfrac{1}{2}(g_{11}{}^1/\omega_1)+\tfrac{3}{8}(g_{11}a_1{}^2/\omega_1)+\tfrac{3}{4}(g_{13}a_2{}^2/\omega_1)]+ \\ &\qquad \tfrac{1}{2}g_{11}{}^2 a_2[\cos(\theta-\psi_2+\psi_1)/a_1(\nu-\omega_2-\omega_1)]\} \end{aligned} \tag{40b}$$

$$\begin{aligned} \dot{a}_2 &= \varepsilon\{\tfrac{1}{2}g_{22}{}^6 a_2-\tfrac{1}{2}a_2[(\partial\omega_2/\partial\tau)/\omega_2]+ \\ &\qquad \tfrac{1}{2}[g_{21}{}^2 a_1\sin(\theta-\psi_2+\psi_1)/(\nu+\omega_1+\omega_2)]\} \end{aligned} \tag{40c}$$

$$\begin{aligned} \dot{\psi}_2 &= \omega_2+\varepsilon\{-[\tfrac{1}{2}(g_{21}{}^1/\omega_2)+\tfrac{3}{8}g_{23}(a_2{}^2/\omega_2)+\tfrac{3}{4}g_{23}(a_1{}^2/\omega_2)]- \\ &\qquad \tfrac{1}{2}g_{21}{}^2 a_1[\cos(\theta-\psi_2+\psi_1)/a_2(\nu+\omega_1+\omega_2)]\} \end{aligned} \tag{40d}$$

The stationary solution is

$$a_1{}^2/a_2{}^2 = -(g_{22}{}^6 g_{11}{}^2\omega_2/g_{11}{}^6 g_{21}{}^2\omega_1) \tag{41a}$$

$$\begin{aligned} \nu &= \omega_2-\omega_1+\varepsilon[\tfrac{1}{2}(g_{11}{}^1/\omega_1)-\tfrac{1}{2}(g_{21}{}^1/\omega_2)+ \\ &\quad a_1{}^2\{(3g_{13}/8\omega_1)-(3g_{23}/4\omega_2)- \\ &\quad [(3g_{13}/4\omega_1)-(3g_{23}/8\omega_2)](g_{11}{}^6 g_{21}{}^2\omega_1/g_{22}{}^6 g_{11}{}^2\omega_2)\}\mp \\ &\quad [-(g_{11}{}^2 g_{21}{}^2/16\omega_1\omega_2)-(g_{11}{}^6 g_{22}{}^6/4)]^{1/2}\times \\ &\qquad \{(g_{22}{}^6/g_{11}{}^6)^{1/2}+(g_{11}{}^6/g_{22}{}^6)^{1/2}\}] \end{aligned} \tag{41b}$$

and the stability conditions are

$$\pm(\partial\nu/\partial a_j) > 0, \quad j = 1,2 \tag{42}$$

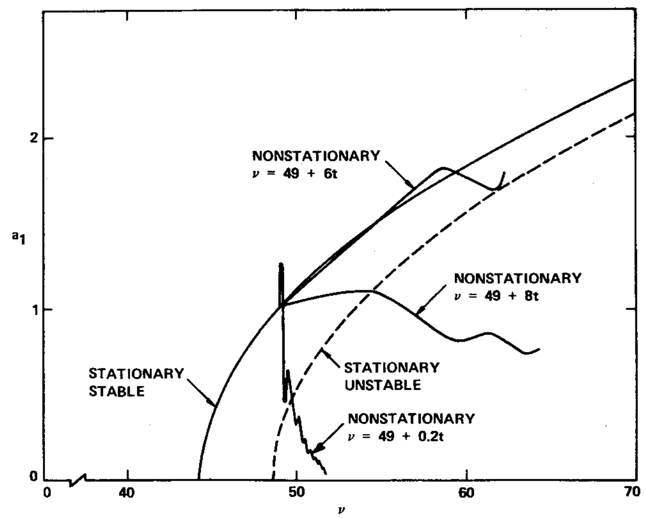

Fig. 4 Nonstationary response for a combination differential resonance, $\nu = \omega_2 - \omega_1$, for linearly increasing frequency of perturbation.

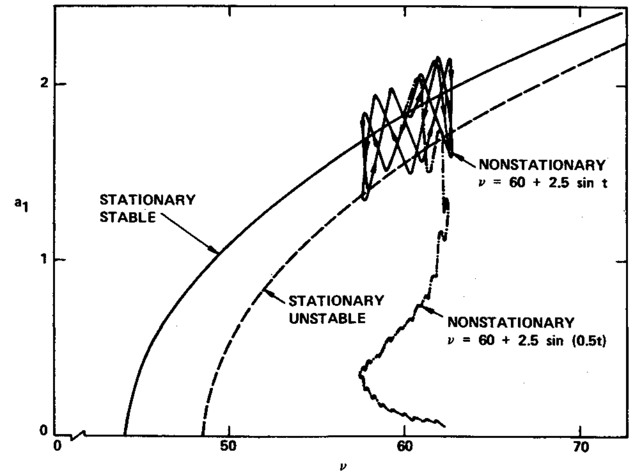

Fig. 5 Nonstationary response for a combination differential resonance, $\nu = \omega_2 - \omega_1$, for periodically varying frequency of perturbation.

Numerical Results

The following parameters have been used for numerical calculation:

$$K = 352, \quad K_1 = 6.25, \quad K_2 = 9.4, \quad I_p/I_1 = 0.0625$$
$$F_1 = 0.128\nu^2 + 0.375, \quad F_2 = 0.128\nu^2 - 0.375$$

The nonstationary responses are obtained by numerically integrating the nonstationary solutions. It should be noted that the natural frequencies and amplitude of the exciting force are functions of ν; i.e., they are time dependent. The stationary solutions of the system and the nonstationary solutions for various functions of ν are plotted for the combination differential resonances $\nu = \omega_2 - 2\omega_1$ and $\nu = \omega_2 - \omega_1$ in Figs. 2 and 3 and Figs. 4 and 5, respectively. It is obvious from the nonstationary response that the rate of the frequency of perturbation, ν, plays a significant role in the modification of the nonstationary response. The nonstationary response may be shifted from one stable solution to another by changing the rate of variation of ν.

Conclusions

The asymptotic method presented in this paper results in a unified approach for the determination of the resonant response of a nonstationary, nonlinear mechanical system for general resonances, including combination resonances. The resonance conditions can be used to determine the possible resonances in a system. The first asymptotic nonstationary solution can be obtained from the general solution, as demonstrated by the calculation of the combination differential resonances $\nu = \omega_2 - 2\omega_1$ and $\nu = \omega_2 - \omega_1$ of the gyroscopic system. The nonstationary responses obtained for various functions of ν indicate that the nonstationary response may shift from one stable mode to another when the rate of variation of the frequency of excitation is changed.

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